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Take up now the case where r_1 and r_2 differ from each other and the difference is not an integer. We may say for convenience that r_1 and r_2 are *like* when they differ by either zero or an integer, and *unlike* when the difference $r_1 - r_2$ is neither zero nor an integer. We have now r_1 and r_2 unlike, and

$$\begin{aligned} r_1 &= \tfrac{1}{2}\left\{(1 - J_1) + \sqrt{(1-J_1)^2 - 4J_2}\right\}, \\ r_2 &= \tfrac{1}{2}\left\{(1 - J_1) - \sqrt{(1-J_1)^2 - 4J_2}\right\}, \\ r_1 - r_2 &= \sqrt{(1-J_1)^2 - 4J_2}, \\ J_1 &= \frac{A_{11} + A_{12} + A_{13}}{2}, \\ J_2 &= \frac{A_{21} + A_{22} + A_{23} + A_{24} + A_{25}}{4}. \end{aligned}$$

The integrals of the differential equation are therefore of the form

$$y_1 = (x - 1)^{r_1} u$$
$$y_2 = (x - 1)^{r_2} v,$$

where

$$u = \sum_{n=0}^{n=\infty} \Delta_n (x-1)^n, \quad v = \sum_{n=0}^{n=\infty} \delta_n (x-1)^n.$$

The differential equation is

$$\frac{d^2y}{dx^2} + \frac{A_{11}x^2 + A_{12}x + A_{13}}{x(1-x^2)} \frac{dy}{dx} + \frac{A_{21}x^4 + A_{22}x^3 + A_{23}x^2 + A_{24}x + A_{25}}{x^2(1-x^2)^2} y = 0;$$

change the variable by the relation

$$x = x' + 1$$

and in the result write for convenience x instead of x': also write

$$A_{11} = -\alpha_1, \ 2A_{11} + A_{12} = -\beta_1, \ A_{11} + A_{12} + A_{13} = -\eta_1,$$
$$A_{21} = -\alpha_2, \ 4A_{21} + A_{22} = -\beta_2, \ 6A_{21} + 3A_{22} + A_{23} = -\gamma_2,$$
$$4A_{21} + 3A_{22} + 2A_{23} + A_{24} = -\zeta_2, \ A_{21} + A_{22} + A_{23} + A_{24} + A_{25} = -\eta_2,$$

and we have

$$\frac{d^2y}{dx^2} = \frac{\alpha_1 x^2 + \beta_1 x + \eta_1}{x(x+1)(x+2)} \frac{dy}{dx} + \frac{\alpha_2 x^4 + \beta_2 x^3 + \gamma_2 x^2 + \zeta_2 x + \eta_2}{x^2(x+1)^2(x+2)^2} y,$$

and also

$$y_1 = x^{r_1} u, \quad u = \sum_{n=0}^{n=\infty} \Delta_n x^n,$$

$$y_2 = x^{r_2} v, \quad v = \sum_{n=0}^{n=\infty} \delta_n x^n.$$

The form of the integral y_1 (*i. e.* the values of the coefficients Δ) having been obtained, it will be only necessary to change r_1 into r_2 in order to find the values of the coefficients δ—that is, in order to find the integral y_2. Writing now $y = x^r u$, we find for u the differential equation

$$\frac{d^2u}{dx^2} = \frac{(\alpha_1 - 2r)x^2 + (\beta_1 - 6r)x + (\eta_1 - 4r)}{x(x+1)(x+2)} \frac{du}{dx} + \frac{1}{x^2(x+1)^2(x+2)^2} \{ [\alpha_2 + r\alpha_1 - r(r-1)]x^4$$
$$+ [\beta_2 + r(\beta_1 + 3\alpha_1) - 6r(r-1)]\,x^3 + [\gamma_2 + r(2\alpha_1 + 3\beta_1 + \eta_1) - 13r(r-1)]\,x^2$$
$$+ [\zeta_2 + r(2\beta_1 + 3\eta_1) - 12r(r-1)]\,x + [\eta_2 + 2r\eta_1 - 4r(r-1)] \}\, u.$$

Multiplying out by $[x(x+1)(x+2)]^2$ we have

$$\{x^6 + 6x^5 + 13x^4 + 12x^3 + 4x^2\} \frac{d^2u}{dx^2}$$
$$= \{(\alpha_1 - 2r)x^5 + (\beta_1 + 3\alpha_1 - 12r)x^4 + (\eta_1 + 3\beta_1 + 2\alpha_1 - 26r)x^3 + (3\eta_1 + 2\beta_1 - 24r)x^2$$
$$+ (2\eta_1 - 8r)\} \frac{du}{dx} + \{[\alpha_2 + r\alpha_1 - r(r-1)]\,x^4 + [\beta_2 + r(\beta_1 + 3\alpha_1) - 6r(r-1)]\,x^3$$
$$+ [\gamma_2 + r(2\alpha_1 + 3\beta_1 + \eta_1) - 13r(r-1)]\,x^2 + [\zeta_2 + r(2\beta_1 + 3\eta_1) - 12r(r-1)]\,x$$
$$+ [\eta_2 + 2r\eta_1 - 4r(r-1)]\}\, u.$$

Substituting for u its value, viz.

$$u = \sum_{n=0}^{n=\infty} \Delta_n x^n,$$

and equating the coefficients of x^{n+1}, we find for the relation connecting five consecutive coefficients Δ

$$\{4n(n+1) - (n+1)(2\eta_1 - 8r) - [\eta_2 + 2r\eta_1 - 4r(r-1)]\}\,\Delta_{n+1}$$
$$+ \{12n(n-1) - n(3\eta_1 + 2\beta_1 - 24r) - [\zeta_2 + r(2\beta_1 + 3\eta_1) - 12r(r-1)]\}\,\Delta_n$$
$$+ \{13(n-1)(n-2) - (n-1)(\eta_1 + 3\beta_1 + 2\alpha_1 - 26r)$$
$$- [\gamma_2 + r(2\alpha_1 + 3\beta_1 + \eta_1) - 13r(r-1)]\}\,\Delta_{n-1}$$
$$+ \{6(n-2)(n-3) - (n-2)(\beta_1 + 3\alpha_1 - 12r)$$
$$- [\beta_2 + r(\beta_1 + 3\alpha) - 6r(r-1)]\}\,\Delta_{n-2}$$
$$+ \{(n-3)(n-4) - (n-3)(\alpha_1 - 2r) - [\alpha_2 + r\alpha_1 - r(r-1)]\}\,\Delta_{n-3} = 0.$$

Computing the coefficients Δ by aid of this formula, replacing x by $x - 1$ and the coefficients α, β, etc., by their values in A_{11}, A_{12} . . . . A_{25}, we have the desired integral of the original differential equation in the region C_1.

Suppose, now, that the roots of the fundamental determinant equations differ by an integer, *i. e.* $r_1 - r_2 =$ (positive) integer. The integrals are of the form $$Y_1 = x^{r_1} \log x, \quad Y_2 = x^{r_2}[v + n \log x].$$

The integral J_1 has its coefficients determined by the last formula. Write

$$v = \sum_{n=0}^{n=\infty} \delta_n x^n$$

and substitute Y_2 in the differential equation; we find (writing r for r_2)

$$\{x^6+6x^5+13x^4+12x^3+4x^2\}\frac{d^2v}{\delta x^2},$$

$$\{(\alpha_1-2r)x^5+(\beta_1+3\alpha_1-12r)x^4+(\gamma_1+3\beta_1+2\alpha_1-26r)x^3+(3\gamma_1+2\beta_1-24r)x^2$$
$$+(2\eta_1-8r)x\}\frac{dv}{\delta x}-\{[\alpha_2+r\alpha_1-r(r-1)]x^4+[\beta_2+r(\beta_1+3\alpha_1)-6r(r-1)]x^3$$
$$+[r_2+r(2\alpha_1+3\beta_1+\gamma_1)-13r(r-1)]x^2+[\zeta_2+r(2\beta_1+3\gamma_1)-12r(r-1)]x$$
$$+[\gamma_2+2r\gamma_1-4r(r-1)]\}v=-[\{2x^5+12x^4+26x^3+24x^2+8x\}\frac{du}{\delta x}$$
$$+\{(2r-1-\alpha_1)x^4+[6(2r-1)-3\alpha_1-\beta_1]x^3+[13(2r-1)-\gamma_1-3\beta_1-2\alpha_1]x^2$$
$$+[12(2r-1)-3\gamma_1-2\beta_1]x+[4(2r-1)-2\gamma_1]\}]n.$$

Equating coefficients of x^{n+1}, we have as the relation connecting the Δ's with the δ's,

$$\{4n(n+1)-(n+1)(2\eta_1-8r)-[\eta_2+2r\eta_1-4r(r-1)]\}\delta_{n+1}$$
$$+\{12n(n-1)-n(3\eta_1+2\beta_1-24r)-[\zeta_2+r(2\beta_1+3\gamma_1)-12r(r-1)]\}\delta_n$$
$$+\{13(n-1)(n-2)-(n-1)(\gamma_1+3\beta_1+2\alpha_1-26r)-[\gamma_2+r(2\alpha_1+3\beta_1+\eta_1)$$
$$-13r(r-1)]\}\delta_{n-1}$$
$$+\{6(n-2)(n-3)-(n-2)(\beta_1+3\alpha_1-12r)-[\beta_2+r(\beta_1+3\alpha_1)-6r(r-1)]\}\delta_{n-2}$$
$$+\{(n-3)(n-4)-(n-3)(\alpha_1-2r)-[\alpha_2+r\alpha_1-r(r-1)]\}\delta_{n-3}$$
$$=-\{[8n+4+8r-2\gamma_1]\Delta_{n+1}+[24n+24r-12-3\gamma_1-2\beta_1]\Delta_n$$
$$+[26n+26r-39-\eta_1-3\beta_1-2\alpha_1]\Delta_{n-1}$$
$$+[12n-30+12r-3\alpha_1-\beta_1]\Delta_{n-2}+[2n-7+2r-\alpha_1]\Delta_{n-3}\}.$$

It will be noticed that the coefficients of δ and Δ in this formula (as in all the preceding ones) are symmetrical in r and n; all the terms in the coefficient of δ_{n+1} which contain only n or r, or both, unite into the one term $4(n+r)(n+r+1)$; the similar terms in the coefficient of δ_n unite into $12(n+r)(n+r-1)$; those in the coefficient of δ_{n-1} into $13(n+r)(n+r-3)$, etc.

(*To be continued.*)